A note on Waring's Problem

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Abstract

In this paper, we will present a new iterative construction for the auxiliary equation of Waring's problem, which seems a little simpler than the one of Vaughan [4], and give a upper bound of $G(k)$, which is same as the one of the paper [4].

Keywords: Waring's Problem, Hardy-Littlewood method, iterative method, auxiliary equation.

1. Introduction

Waring's problem is a well-known problem in number theory, the original statement is that, for each natural number k, there is a positive integer $g(k)$, such that each natural numbers may be represented a sum of at most $g(k)$ kth powers of natural numbers. Since the original problem was proved by Hilbert, and the values of $g(k)$ have been almost completely known, the modern version of the problem is to find the least integer $G(k)$ such that each sufficient large integer may be represented a sum of at most $G(k)$ kth powers of natural numbers. In about 1920's, Hardy, Ramanujan and Littlewood proposed so called circle method which at first appears hope in solving some problems of number theory, since then many mathematicians such as Vinogradov, Davenport, Hua, Chen, Vaughan, Wooley and others have made enduring and great effort on exploiting Hardy-littlewood circle method, and have succeed in getting plenty good results for $G(k)$. For the details is referred to see the Vaughan and Wooley's survey paper [6].

Suppose that N is a sufficient large number, for a given integer k, denoted by $P = \left[N^{1/k}\right]$. For s integer sets $X_i, 1 \le i \le s, X_i \subseteq [0, P]$, denoted by $\mathfrak{A} = \left\{ \sum_{1\le i\le s} x_i^k \mid x_i \in X_i, 1 \le i \le s \right\}$.

Let $\gamma(m)$ be the number of occurrences of number m in $\mathfrak{A}$, that is, the characteristic function of set $\mathfrak{A}$. It is clear that $\prod_{1\le i\le s} |X_i| = \sum_m \gamma(m)$, hence

$$\left(\prod_{1\le i\le s} |X_i|\right)^2 = \left(\sum_m \gamma(m)\right)^2 \le \sum_{\gamma(m)>0} 1 \sum_m \gamma(m)^2$$

So,

$$\sum_{\gamma(m)>0} 1 \ge \left(\prod_{1\le i\le s} |X_i|\right)^2 \Big/ \sum_m \gamma(m)^2 .$$

The formula above indicates that $\sum_m \gamma(m)^2$ is more less, the size $|\mathfrak{A}|$ is more large when $\prod_{1\le i\le s} |X_i|$ given. In Hardy-littlewood method, an important work is to find as possible as least s and sets $X_i, 1 \le i \le s,$ such that $\sum_m \gamma(m)^2$ as possible as small and that set $\mathfrak{A}$ is dense in $[1, N]$ with some sense. Let $X = (X_1, X_2, \cdots, X_s)$, $S_s(P, X) = \sum_m \gamma(m)^2$, it is clear that

$S_s(P,X)$ is the number of solutions of equation

$$x_1^k+\cdots+x_s^k=y_1^k+\cdots+y_s^k,\quad x_i,y_i\in X_i,1\le i\le s.$$

The equation above is called auxiliary equation of Waring problem.

Earlier work such as Vinogradov [7] and Davenport [1,2], the selection of sets $X_i,1\le i\le s,$ are different from each other. Vaughan [4] provided a creative iterative method with the sets X_i are same each other, $X_i=\mathscr{A}(P,R),1\le i\le s,$ where $\mathscr{A}(P,R)$ is a set of so called "smooth" integers. In their papers, $S_s(P,X)$ is written as $S_s(P,R)$. It is not difficult to imagine that there is a merit for such homogeneous construction is that it may be represented as a simple integral form, which is favorable to be varied by Hölder inequality, this feature is reflected very well in the paper [4].

In this paper, we will present a new construction for domain set X of the auxiliary equation, which is seemingly a little simpler than the one of "smooth" numbers in [4] and [8], but most arguments and results are almost same as ones in [4]. With the new homogeneous construction, we also obtain

Theorem 1. For sufficient large k,

$$G(k)\le 2k\left(\log(k\log k)+1+\log 2+O\left(\frac{\log\log k}{\log k}\right)\right).\qquad (1)$$

2. The Proof of Theorem 1.

P is sufficient great, θ is a constant, with that $\theta>1/k$. Let $\tilde{P}=P^{1+\theta}$, $\mathscr{P}$ is a set of prime numbers p in interval $[P^\theta/2,P^\theta]$, write $|\mathscr{P}|=Z,$ then we know $Z\doteq P^\theta/2\log(P^\theta).$ Define recursively

$$\mathscr{C}(\tilde{P})=\{x\cdot p\mid x\in\mathscr{C}(P),p\in\mathscr{P}\}.\qquad (2)$$

In this paper, we simply write $S_s(P)=S_s(P,X)$ as $X=\mathscr{C}(P)$.

Lemma 1.

$$S_s(\tilde{P})\ll Z^sS_s(P)+Z^{2s}PS_{s-1}(P).\qquad (3)$$

Proof. As usual, write $e(x)=e^{2\pi ix}$, let

$$f(\alpha)=\sum_{x\in\mathscr{C}(P)} e\left(x^k\alpha\right),\ \ f(\alpha,p)=\sum_{x\in\mathscr{C}(P)} e\left(p^k x^k\alpha\right),\ \ \tilde{f}(\alpha)=\sum_{y\in\mathscr{C}(\tilde{P})} e\left(y^k\alpha\right).$$

Then clearly,

$$\tilde{f}(\alpha)=\sum_{p\in\mathscr{P}}\sum_{x\in\mathscr{C}(P)} e\left(p^k x^k\alpha\right)=\sum_{p\in\mathscr{P}} f(\alpha,p)\,.$$

Applying Hölder's inequality, it has

$$S_s(\tilde{P})=\int_0^1\left|\tilde{f}(\alpha)^{2s}\right|d\alpha=\int_0^1\left|\tilde{f}(\alpha)^s\,\tilde{f}(\alpha)^{-s}\right|d\alpha$$

$$=\int_0^1\left(\sum_{p\in\mathscr{P}}\sum_{q\in\mathscr{P}} f(\alpha,p)\,f(-\alpha,q)\right)^s d\alpha$$

$$=\int_0^1\left(\sum_{p=q}+\sum_{p\neq q}\right)^s d\alpha$$

$$\ll\int_0^1\left(Z\cdot\left|f(\alpha)\right|^2\right)^s d\alpha+\int_0^1\Big(\sum_{p,q\in\mathscr{P},\ p\neq q}\left|f(\alpha,p)\,f(-\alpha,q)\right|\Big)^s d\alpha$$

$$\ll Z^s S_s(P)+Z^{2(s-1)}\sum_{p,q\in\mathscr{P},\ p\neq q}\int_0^1\left|f(\alpha,p)\,f(\alpha,q)\right|^s d\alpha.$$

Moreover, let $\Lambda(\alpha,p,q)=f(\alpha,p)^{s-1}f(\alpha,q)$, then by Cauchy inequality, it has

$$\int_0^1\left|f(\alpha,p)^s\,f(\alpha,q)^s\right|d\alpha=\int_0^1\left|\Lambda(\alpha,p,q)\right|\left|\Lambda(\alpha,q,p)\right|d\alpha$$

$$\leq\left(\int_0^1\left|\Lambda(\alpha,p,q)\right|^2 d\alpha\right)^{1/2}\left(\int_0^1\left|\Lambda(\alpha,q,p)\right|^2 d\alpha\right)^{1/2}$$

And

$$\sum_{p,q\in\mathscr{P},\,p\neq q}\int_0^1\left|f(\alpha,p)^s\,f(\alpha,q)^s\right|d\alpha\leq\sum_{p,q\in\mathscr{P},\,p\neq q}\left(\int_0^1\left|\Lambda(\alpha,p,q)\right|^2 d\alpha\right)^{1/2}\left(\int_0^1\left|\Lambda(\alpha,q,p)\right|^2 d\alpha\right)^{1/2}$$

$$\leq\sum_{p,q\in\mathscr{P},\,p\neq q}\int_0^1\left|\Lambda(\alpha,p,q)\right|^2 d\alpha$$

Denoted by $T_{p,q}$ the inner integral, which is the number of solutions of equation

$$p^k\left((x_1^k+x_2^k+\cdots+x_{s-1}^k)-(y_1^k+y_2^k+\cdots+y_{s-1}^k)\right)=q^k\left(y^k-x^k\right)\qquad(4)$$

With $x,y\in\mathscr{C}(P),\,x_i,y_i\in\mathscr{C}(P),1\leq i\leq s-1,\ \ p,q\in\mathscr{P},\ p\neq q.$

Therefore, it must be that $y^k\equiv x^k \mod p^k$. We know that there are at most finite solutions for the congruence $x^k\equiv m \mod p^k$, so we may divide residue system into finite classes such that there is at most one solution in each class. This means in one class $y^k\equiv x^k \mod p^k$ implies $y\equiv x \mod p^k$, and so here $x=y$, for $\theta>1/k$. Hence

$T_{p,q} \ll PS_{s-1}(P)$,

and

$$\sum_{p,q\in\mathscr{P},\, p\neq q} \int_0^1 \left| f(\alpha,p) f(\alpha,q) \right|^s d\alpha \ll Z^2 PS_{s-1}(P).$$

And then,

$$S_s(\tilde{P}) \ll Z^s S_s(P) + Z^{2s} PS_{s-1}(P) . \qquad \square$$

Corollary 1. Suppose that $S_s(P) = P^{\lambda_s}$, then for $s \geq 2$,

$$\lambda_s \leq (2s-k) + (k-2)\left(\frac{k}{k+1}\right)^{s-2} . \tag{5}$$

Proof. By Lemma 1, it has

$$\tilde{P}^{\lambda_s} \ll Z^s P^{\lambda_s} + Z^{2s} P^{\lambda_{s-1}+1}.$$

As $\lambda_s \geq s$, and $Z \doteq P^{\theta} /(2\theta \log P)$, so the first term of the right-side of the inequality above is sub-term may be omitted, and hence

$$P^{(1+\theta)\lambda_s} \ll P^{\lambda_{s-1}+1+2s\theta}.$$

i.e.

$$\lambda_s \leq \frac{\lambda_{s-1}}{(1+\theta)} + \frac{(1+2s\theta)}{(1+\theta)} .$$

From the recursive inequality, with a simple calculate, it follows

$$\lambda_s \leq 2s - \frac{1}{\theta} + \left(\frac{1}{\theta} - 2\right)\left(\frac{1}{1+\theta}\right)^{s-2}$$

Let $\theta \to 1/k$, (5) is followed. $\square$

Let $\tau = 2kP^{k-1}$, $I = [\tau^{-1}, 1+\tau^{-1}]$. For integers a, q , denotes

$$\mathfrak{M}(q,a) = \left\{ \alpha \,\middle|\, \left|\alpha - a/q\right| \leq 1/q\tau \right\} ,$$

And for positive number W $(\leq P)$, denotes

$$\mathfrak{N}(q,a) = \left\{ \alpha \,\middle|\, \left|\alpha - a/q\right| \leq W/(q\tau P) \right\} .$$

$\mathfrak{M}$ is the union of the $\mathfrak{M}(q,a)$ with $1 \leq a \leq q \leq P$, $(a,q) = 1$. $\mathfrak{m} = I \setminus \mathfrak{M}$.

$\mathfrak{N}$ is the union of the $\mathfrak{N}(q,a)$ with $1 \leq a \leq q \leq W$, $(a,q) = 1$. $\mathfrak{n} = I \setminus \mathfrak{N}$.

Let $f(\alpha)=\sum_{x\le P} e(\alpha x^k)$, $g(\alpha)=\sum_{x\in\mathscr{C}(P)} e(\alpha x^k)$.

The proof of Theorem 1 will require the following results.

Lemma 2. For $\alpha\in\mathfrak{m}$,

$$f(\alpha)\ll P^{1-1/2^{k-1}}. \tag{6}$$

Lemma 3. Suppose that $k\ge 3, s\ge k+2,$ then

$$\int_{\mathfrak{M}}\left|f(\alpha)^s\right|d\alpha \ll P^{s-k}, \tag{7}$$

$$\int_{\mathfrak{M}\setminus\mathfrak{N}}\left|f(\alpha)^s\right|d\alpha \ll W^{\varepsilon-1/k}P^{s-k}. \tag{8}$$

Lemma 4. Let

$$V(\alpha)=\sum_{x/2<p\le X}\sum_{y\le Y} b_y e(yp^k\alpha),$$

where b_y are arbitrary complex numbers. Suppose that $\alpha=a/q+\beta,$ with $|\beta|\le \frac{1}{2}q^{-1}X^{-k}$,

$q\le 2X^k, (a,q)=1$, that $Y\gg X^k$, and that when $q\le X$ one has $|\beta|\gg q^{-1}X^{1-k}Y^{-1}$. Then

$$V(\alpha)\ll\left(XY^{1+\varepsilon}\sum_{y\le Y}\left|b_y\right|^2\right)^{1/2}. \tag{9}$$

(6) is from Weyl' inequality, Lemmas 3 and 4 are just Lemma 5.4 of [4] and Lemma 5.1 of [3] respectively, the proofs are referred to see [3], [4], or [7].

Lemma 5. Let $X=P^{1/2}$, $h(\alpha)=\sum_{(X/2)\le p\le X}\sum_{x\in\mathscr{C}(X)} e(\alpha p^k x^k)$. Then

$$|h(\alpha)|\ll P^{1-\hat{\sigma}}, \quad \text{for } \alpha\in\mathfrak{n}. \tag{10}$$

where $\hat{\sigma}=\dfrac{\log(1+1/k)}{4(1+\lambda)}$, $\lambda=\log k+\log\log k+O\left(\dfrac{\log\log k}{\log k}\right)$.

Proof. By Hölder inequality, it has

$$|h(\alpha)|^{2s}\le X^{2s-1}\sum_{(X/2)\le p\le X}\left|\sum_{y\in Y} b_y e(\alpha p^k y)\right|^2$$

Where $Y=sX^k$, and b_y is the number of solutions of

$$x_1^k+x_2^k+\cdots+x_s^k=y, \quad x_i\in\mathscr{C}(X), 1\le i\le s.$$

When $\alpha \in \mathfrak{n}$, by lemma 4 and (5), it has

$$|h(\alpha)|^{2s} \ll X^{2s-1+k+\lambda_s} \le X^{4s-1+(k-2)(k/(k+1))^{s-2}}$$

That is,

$$|h(\alpha)| \ll P^{(4s-1+(k-2)(k/(k+1))^{s-2})/4s} = P^{1-(1-(k-2)(k/(k+1))^{s-2})/4s} = P^{1-\sigma}$$

Where $\sigma = \sigma(k,s) = (1-(k-2)(k/(k+1))^{s-2})/4s$

From calculus, we know that when $s = \lambda / \log(1+1/k)$, $\sigma(k,s)$ arrives the maximum value

$$\hat{\sigma} = \frac{\log(1+1/k)}{4(1+\lambda)},$$

Where λ is the root of equation $(1+\lambda)\beta = e^{\lambda}$, $\beta = (k-2)(k+1)^2/k^2$. It is easy to know

$$\lambda = \log k + \log\log k + O\left(\frac{\log\log k}{\log k}\right). \qquad \square$$

Suppose that u and t are two natural numbers, with that $u \ge k+1$, $2t\hat{\sigma} \ge (k-2)\left(\frac{k}{k+1}\right)^{u-2}$,

define

$$L = \int_0^1 \left| f(\alpha)^2 g(\alpha)^{2u} h(\alpha)^{2t} \right| d\alpha, \quad I = \int_{\mathfrak{n}} \left| f(\alpha) g(\alpha)^{2u} h(\alpha)^{2t} \right| d\alpha.$$

Lemma 6.

$$L \ll P^{2+2t+2u-k}, \quad I \ll P^{2+2t+2u-k} W^{-\delta}. \tag{11}$$

Proof.

$$L = \int_{\mathfrak{m}} \left| f(\alpha)^2 g(\alpha)^{2u} h(\alpha)^{2t} \right| d\alpha + \int_{\mathfrak{M}} \left| f(\alpha)^2 g(\alpha)^{2u} h(\alpha)^{2t} \right| d\alpha.$$

By Hölder inequality and Lemmas 1, 5, it has

$$\int_{\mathfrak{m}} \left| f(\alpha)^2 g(\alpha)^{2u} h(\alpha)^{2t} \right| d\alpha \ll P^{2+2t(1-\hat{\sigma})} \int_{\mathfrak{m}} \left| g(\alpha)^{2u} \right| d\alpha$$

$$\ll P^{2+2t(1-\hat{\sigma})} P^{2u-k+(1-1/k)^{u-2}} \ll P^{2+2t+2u-k}.$$

$$\int_{\mathfrak{M}} \left| f(\alpha)^2 g(\alpha)^{2u} h(\alpha)^{2t} \right| d\alpha \le \left(\int_{\mathfrak{M}} \left| f(\alpha)^{k+2} \right| d\alpha \right)^{2/(k+2)} \left(\int_0^1 \left| g(\alpha)^{2u} h(\alpha)^{2t} \right|^{(k+2)/k} d\alpha \right)^{k/(k+2)}$$

$$\int_0^1 \left| g(\alpha)^{2u} h(\alpha)^{2t} \right|^{(k+2)/k} d\alpha = \int_0^1 \left| g(\alpha)^{2u+4u/k} h(\alpha)^{2t+4t/k} \right| d\alpha$$

$$= \int_0^1 \left| g(\alpha)^{2+2u} h(\alpha)^{2t} \right| \left| g(\alpha)^{4u/k-2} h(\alpha)^{4t/k} \right| d\alpha$$

$$\le P^{4u/k-2+4t/k}\int_0^1\left|g(\alpha)^{2u+2}h(\alpha)^{2t}\right|d\alpha$$
$$\le P^{4u/k-2+4t/k}\int_0^1\left|g(\alpha)^{2u+2}h(\alpha)^{2t}\right|d\alpha$$

Considering underlying equation, we can know

$$\int_0^1\left|g(\alpha)^{2u+2}h(\alpha)^{2t}\right|d\alpha\le\int_0^1\left|f(\alpha)^2g(\alpha)^{2u}h(\alpha)^{2t}\right|d\alpha=L.$$

In addition, by (8), it has $\int_{\mathfrak{M}}\left|f(\alpha)^{k+2}\right|d\alpha\ll P^2$. So

$$L\ll P^{2+2t+2u-k}+P^{2\times 2/(k+2)}\cdot\left(LP^{(4u-2k+4t)/k}\right)^{k/(k+2)}$$
$$\ll P^{2+2t+2u-k}+P^{2\times 2/(k+2)}\cdot L^{k/(k+2)}\left(P^{(4u-2k+4t)/(k+2)}\right).$$

And,

$$L\ll P^{(2+2t+2u-k)}.$$

Similarly,

$$I=\int_{\mathfrak{n}}\left|f(\alpha)g(\alpha)^{2u}h(\alpha)^{2t}\right|d\alpha$$
$$=\int_{\mathfrak{m}}\left|f(\alpha)g(\alpha)^{2u}h(\alpha)^{2t}\right|d\alpha+\int_{\mathfrak{M}\setminus\mathfrak{N}}\left|f(\alpha)g(\alpha)^{2u}h(\alpha)^{2t}\right|d\alpha$$
$$\ll P^{1+2u+2t-k-\delta}+\left(\int_{(\mathfrak{M}\setminus\mathfrak{N})}\left|f(\alpha)\right|^{k+2}d\alpha\right)^{1/(k+2)}\left(\int_{(\mathfrak{M}\setminus\mathfrak{N})}\left|g(\alpha)^{2u}h(\alpha)^{2t}\right|^{(k+2)/(k+1)}d\alpha\right)^{(k+1)/(k+2)}$$
$$\ll P^{1+2u+2t-k-\delta}+\left(\int_{(\mathfrak{M}\setminus\mathfrak{N})}\left|f(\alpha)\right|^{k+2}d\alpha\right)^{1/(k+2)}\left(\int_{(\mathfrak{M}\setminus\mathfrak{N})}\left|g(\alpha)^{2u}h(\alpha)^{2t}\right|^{(k+2)/(k+1)}d\alpha\right)^{(k+1)/(k+2)}$$

$$\int_{(\mathfrak{M}\setminus\mathfrak{N})}\left|g(\alpha)^{2u}h(\alpha)^{2t}\right|^{(k+2)/(k+1)}d\alpha\le\int_0^1\left|g(\alpha)^{2u}h(\alpha)^{2t}\right|^{(k+2)/(k+1)}d\alpha$$
$$\le LP^{2u/(k+1)+2t/(k+1)-2}$$
$$I\ll P^{1+2u+2t-k-\delta}+\left(P^2W^{-\delta}\right)^{1/(k+2)}\left(LP^{2u/(k+1)+2t/(k+1)-2}\right)^{(k+1)/(k+2)}$$

It follows

$$I\ll P^{2+2t+2u-k}W^{-\delta}.$$ □

On the other hand, by elementary number theory, it is known the distribution of prime numbers in a larger interval is asymptotically equal $\bmod q$. Explicitly, denoted by $\pi(x)$ and $\pi(c,d)$ are the numbers of prime numbers in intervals $[0,x]$ and $[c,d]$ respectively. For $0<a<q\le P$, $(a,q)=1$, and $\Delta=[c,d]$, let

$$\pi(x;q,a)=\sum_{\substack{p\equiv a \bmod q\\ p\le x}}1,\quad \pi(\Delta;q,a)=\sum_{\substack{p\equiv a \bmod q\\ c\le p\le d}}1,\quad \zeta(x;q,a)=\sum_{\substack{t\equiv a \bmod q\\ t\in\mathscr{C}(x)}}1.$$

Then

$$\pi(x;q,a)=(1+o(1))\frac{\pi(x)}{\varphi(q)}, \qquad \pi(\Delta;q,a)=(1+o(1))\frac{\pi(c,d)}{\varphi(q)}.$$

From formulas above, it is easy to infer that

$$\zeta(x;q,a)=(1+o(1))\frac{|\mathscr{C}(x)|}{\varphi(q)}. \tag{12}$$

This indicates that the contribution from $\mathfrak{N}$ of $g(\alpha)$ and $h(\alpha)$ may be treated in the usual way as for the sequence of integers $\bmod q$ but with a proportion of $|\mathscr{C}(P)|/P$. Besides, by a simply calculate, it is easy to know

$$|\mathscr{C}(P)|\doteq\frac{P}{\log(P)^{(\eta+1)/2}}\cdot\left(\frac{k+1}{2}\right)^{\eta}, \tag{13}$$

Where $\eta=k\log\log P$. So, $|\mathscr{C}(P)|>P^{1-\varepsilon}$. Hence, we have

$$\int_{\mathfrak{N}} f(\alpha)g(\alpha)^{2u}h(\alpha)^{2t}e(-N\alpha)d\alpha \gg P^{1+2u+2t-k}.$$

and

$$R(n)=\int_0^1 f(\alpha)g(\alpha)^{2u}h(\alpha)^{2t}e(-N\alpha)d\alpha \gg P^{1+2u+2t-k}.$$

Take $v=u-2, t=1+\left[\frac{k-2}{2\hat{\sigma}}\left(\frac{k}{k+1}\right)^{v}\right]$, it follows

$$G(k)\le 7+2v+2\left[\frac{k-2}{2\hat{\sigma}}\left(\frac{k}{k+1}\right)^{v}\right].$$

From calculus, we can know that the right-side of the above arrives the maximum as v close to $\log(\mu(k-2)/2\hat{\sigma})/\mu, \quad \mu=\log((k+1)/k)$. Hence,

$$G(k)\le 7+2\log(\mu(k-2)/2\hat{\sigma})/\mu+2\left[\frac{1}{\mu}\right]$$

$$\le 2k\left(\log(k\log k)+1+\log 2+O\left(\frac{\log\log k}{\log k}\right)\right).$$

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